

Transverse Fluctuations in the Driven Lattice Gas

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We define a transverse correlation length suitable to discuss the finite-size-scaling behavior of an out-of-equilibrium lattice gas, whose correlation functions decay algebraically with the distance. By numerical simulations we verify that this definition has a good infinite-volume limit independent of the lattice geometry. We study the transverse fluctuations as they can select the correct field-theoretical description. By means of a careful finite-size scaling analysis, without tunable parameters, we show that they are Gaussian, in agreement with the predictions of the model proposed by Janssen, Schmittmann, Leung, and Cardy.

While the statistical mechanics of systems in thermal equilibrium is well established, no sound framework is available for nonequilibrium systems, although some interesting results have been recently obtained [1]. At present, most of the research focuses on very specific models. Among them, the driven lattice gas (DLG), introduced by Katz, Lebowitz, and Spohn [2], has attracted much attention since it is one of the simplest nontrivial models with a nonequilibrium steady state [3]. The DLG is a generalization of the lattice gas. One considers a hypercubic lattice and for each site x introduces an occupation variable n_x , which can be either zero (empty site) or one (occupied site). Then, one introduces an external field E along a lattice direction and a generalization of the Kawasaki dynamics for the lattice gas with nearest-neighbor interactions. In practice, one randomly chooses a lattice link $\langle xy \rangle$, and, if $n_x \neq n_y$, proposes a particle jump which is accepted with Metropolis probabilities $w(\beta\Delta H + \beta E\ell)$, where $\ell = (1, 0, -1)$ for jumps (along, transverse, opposite) to the field direction, $w(x) = \min(1, e^{-x})$, and ΔH is the variation of the standard lattice-gas nearest-neighbor interaction

$$H = -4 \sum_{\langle xy \rangle} n_x n_y. \quad (1)$$

As usual, the parameter β plays the role of an inverse temperature. A nontrivial dynamics is obtained by considering periodic boundary conditions in the field direction. Indeed, in this case a particle current sets in, giving rise to a nonequilibrium stationary state that is non-Gibbsian.

At half filling, the DLG undergoes a second-order phase transition. Indeed, at high temperatures the steady state is disordered while at low temperatures the system orders: the particles condense forming a strip parallel to the field direction. The two temperature regions are separated by a phase transition occurring at the critical value $\beta_c(E)$ depending on the field E . $\beta_c(E)$ converges to the Ising critical value β_I for $E \rightarrow 0$, and, interestingly enough, increases with E , and saturates at a finite value $\beta_c(\infty)$ when E diverges. Such a transition is different in nature from the order/disorder one occurring in the lattice gas. For instance, it is *strongly anisotropic*, i.e. fluctuations in density correlation functions behave in a qualitatively different way depending on whether one considers points that belong to lines that are parallel or orthogonal to the field E .

Some time ago Janssen, Schmittmann, Leung, and Cardy [4] (JSLC) proposed a Langevin equation for the coarse-grained density field, which incorporates the main features of the DLG, i.e. a conserved dynamics and the anisotropy induced by the external field and should therefore describe the critical behavior of the DLG. By means of a standard Renormalization-Group (RG) analysis, critical exponents were exactly computed in generic dimension d for $2 < d < d_c = 5$. This model predicts a strongly anisotropic behavior with correlations that increase with different exponents ν_{\parallel} and ν_{\perp} in the directions parallel and orthogonal to the external field [3]. The corresponding anisotropy exponent $\Delta = (\nu_{\parallel}/\nu_{\perp}) - 1$ can be exactly computed finding $\Delta = (8 - d)/3$. Another re-

markable property of the JSLC model is that transverse fluctuations become Gaussian in the critical limit. Extensive numerical simulations [5] have confirmed many predictions of the JSLC model, although some notable discrepancies still remain. The conclusions of JSLC have been recently questioned by Garrido, de los Santos, and Muñoz [6]. They argued that the DLG for $E = \infty$ is not described by the JSLC model but is rather in the same universality class as the Randomly Driven Lattice Gas (RDLG). This model has upper critical dimension $d_c = 3$. The RG analysis done in Ref. [7] leads to series expansions for the critical exponents in powers of $\epsilon \equiv d_c - d$. In particular, $\Delta = 1 + O(\epsilon^2)$, consistent with $\Delta \approx 1$ in $d = 2$ (the case we will consider in our numerical simulations), quite different from the JSLC prediction $\Delta = 2$. A second notable difference is that in the RDLG transverse critical fluctuations are not Gaussian. Apparently, numerical simulations of the DLG [8] are also in agreement with the RDLG scenario and consistent with $\Delta \approx 1$ in two dimensions.

In view of these contradictory recent results, new numerical investigations are necessary, in order to decide which of these two models really describes the DLG universality class. In this paper, we will focus on the transverse fluctuations for the very simple reason that in the JSLC model there is a very strong prediction: transverse correlation functions are Gaussian so that, beside critical exponents, one can also compute exactly the scaling functions of several observables. Therefore, it is possible to make stronger tests of the JSLC model.

In order to test the JSLC predictions, we will investigate the Finite-Size Scaling (FSS) behavior of several observables. For an isotropic model on a lattice L^d , the FSS limit corresponds to $t \equiv 1 - \beta/\beta_c \rightarrow 0$, $L \rightarrow \infty$, keeping $tL^{1/\nu}$ constant. This has to be modified for strongly anisotropic models [9] like the DLG. It has been argued that, for a geometry $L_{\parallel} \times L_{\perp}^{d-1}$, one has to keep fixed both combinations $tL_{\parallel}^{1/\nu_{\parallel}}$ and $tL_{\perp}^{1/\nu_{\perp}}$, and therefore also the so-called *aspect ratio* $S_{\Delta} = L_{\parallel}^{1/(1+\Delta)}/L_{\perp}$. Then, if an observable \mathcal{O} diverges at criticality as $t^{-z_{\mathcal{O}}}$, in a finite lattice one has

$$\mathcal{O}(\beta; L_{\parallel}, L) \approx L^{z_{\mathcal{O}}/\nu_{\perp}} f_{\mathcal{O}}(t^{-\nu_{\perp}}/L, S_{\Delta}), \quad (2)$$

where we have neglected subleading scaling corrections. To simplify the notation we write L instead of L_{\perp} and, below, ξ_L for the *transverse* correlation length. In many numerical studies Eq. (2) has been tested for several observables, but this can be a very weak test since several parameters, β_c , ν_{\perp} , $z_{\mathcal{O}}$, and Δ , must be tuned in order to fit the numerical data. A stronger FSS test can be performed if one uses a suitably defined correlation length. In this case, using a transverse (infinite volume) correlation length ξ_{∞} , Eq. (2) may be written in the form

$$\mathcal{O}(\beta; L_{\parallel}, L) \approx L^{z_{\mathcal{O}}/\nu_{\perp}} \tilde{f}_{\mathcal{O}}(\xi_{\infty}(\beta)/L, S_{\Delta}). \quad (3)$$

In this equation β_c does not explicitly appear and $z_{\mathcal{O}}$ and ν_{\perp} enter only through their ratio, cancelling when one looks directly at the correlation length. One can also eliminate this unknown by considering the ratio of the observables on two different lattices with transverse sizes L and αL . In this case we have

$$\frac{\mathcal{O}(\beta; \alpha^{1+\Delta} L_{\parallel}, \alpha L)}{\mathcal{O}(\beta; L_{\parallel}, L)} \approx F_{\mathcal{O}} \left(\frac{\xi(\beta; L_{\parallel}, L)}{L}, S_{\Delta}, \alpha \right), \quad (4)$$

where only Δ has to be fixed *a priori*.

In order to test Eqs. (3) and (4), one has to define a finite-size correlation length. In the DLG this is not obvious because correlation functions always decay algebraically at large distances [10]. A parallel correlation length was defined in Ref. [11], but it suffers from many ambiguities (see the discussion in Ref. [3]). The definition of a transverse correlation length is even more difficult, because of the presence of negative correlations at large distances [3, 5].

In this paper we propose a new definition that generalizes the second-moment correlation length used in equilibrium systems. The basic observation is that the infinite-volume transverse two-point *wall-wall* correlation function decays exponentially, so that a transverse correlation length can be naturally defined in the thermodynamic limit. The extension of this definition to finite volumes requires some care because of the conserved dynamics, which makes the two-point function vanish at zero momentum. Here, we will use the results of Ref. [12]. Given the Fourier transform $\tilde{G}(q)$ of the two-point correlation function $\langle n_x n_0 \rangle$, we focus on the transverse correlation $\tilde{G}_{\perp}(q) \equiv \tilde{G}(\{q_{\parallel} = 0, q_{\perp} = q\})$ and define

$$\xi_{ij} \equiv \sqrt{\frac{1}{\hat{q}_j^2 - \hat{q}_i^2} \left(\frac{\tilde{G}_{\perp}(q_i)}{\tilde{G}_{\perp}(q_j)} - 1 \right)}, \quad (5)$$

where $\hat{q}_n = 2 \sin(\pi n/L)$ is the lattice momentum. If the infinite-volume transverse wall-wall correlation function decays exponentially, $\tilde{G}_{\perp}(q)$ has a regular expansion in powers of q^2 and

$$\tilde{G}_{\perp}^{-1}(q) \approx \chi_{\infty}^{-1} [1 + b^2 q^2 + O(q^4)], \quad (6)$$

where the coefficient b of q^2 naturally defines a transverse correlation length ξ_{∞} . We expect Eq. (6) to hold also in a finite box. Then, starting from Eq. (5), it is easy to show that ξ_{ij} converges to ξ_{∞} as $L \rightarrow \infty$, justifying our definition of finite-volume correlation length. In the subsequent analysis we consider ξ_{13} as the finite-volume (transverse) correlation length $\xi_L(T) \equiv \xi(T; S_{\Delta}^{1+\Delta} L^{1+\Delta}, L)$. As in previous studies, we also define a finite-volume transverse susceptibility as $\chi_L = \tilde{G}_{\perp}(2\pi/L)$.

As we already said, in the JSLC model transverse fluctuations are Gaussian. This allows us to compute the

scaling function appearing in Eq. (4) for ξ_L and χ_L . If $\tilde{G}_\perp(q)$ is Gaussian we have

$$F_\xi(z, S_2, \alpha) = [1 - (1 - \alpha^{-2})(2\pi)^2 z^2]^{-1/2}, \quad (7)$$

with $z \equiv \xi_L/L$ and $F_\chi(z, S_2, \alpha) = F_\xi(z, S_2, \alpha)^2$. We can also compute $\tilde{f}_\xi(x, S_2)$, see Eq. (3), obtaining

$$\tilde{f}_\xi(x, S_2) = (4\pi^2 + 1/x^2)^{-1/2}. \quad (8)$$

In this paper we want to make a high-precision test of the theoretical predictions of the JSLC model, Eqs. (7) and (8). We work in two dimensions at infinite driving field. Since we wish to test the JSLC predictions we fix $\Delta = 2$ and consider lattice sizes with $S_2 \approx 0.200$. The largest lattice corresponds to $L_\parallel = 884$, $L = 48$. For each lattice size, we compute χ_L and ξ_L for several values of β lying between 0.28 and 0.312.

As a preliminary test, we verify that our definition of ξ_L has a good thermodynamic limit. For this purpose, we introduce the following quantity (the reason for this definition will be presented below)

$$\tau_L(\beta) \equiv \xi_L^{-2}(\beta) - 4\pi^2 L^{-2}. \quad (9)$$

In Fig. 1 we plot $\tau_L(\beta)$ versus $1/L^2$ at several inverse temperatures β . For each β , $\tau_L(\beta)$ converges to a finite constant, showing that our definition has a finite infinite-volume limit. Moreover, the same result is obtained by using sequences of lattices with S_2 or S_1 fixed: the result does not depend on the way in which L_\parallel and L go to infinity. As expected, when the temperature approaches the critical value, it is necessary to use larger and larger lattices to see the convergence to the infinite-volume limit. At β fixed we expect the convergence to become eventually exponential in L . However, for lattices with S_2 fixed we observed an intermediate region of values of L in which τ_L is apparently constant. Such a region widens as β approaches the critical point and is therefore in excellent agreement with the relation

$$\xi_\infty^{-2}(\beta) \approx \tau_L(\beta) = \xi_L^{-2}(\beta) - 4\pi^2 L^{-2} \quad (10)$$

in the FSS limit $L \rightarrow \infty$, $\beta \rightarrow \beta_c$, neglecting subleading corrections that will depend in general on the chosen value of S_2 and are particularly small for our choice $S_2 \approx 0.200$. Eq. (10) immediately gives Eq. (8). Thus, the results presented on Fig. 1 are perfectly consistent with $\Delta = 2$ and the JSLC prediction for $\tilde{f}_\xi(x, S_2)$.

We want now to make a precise test of Eq. (7). In Fig. 2 we report the results of our simulations for the ratio ξ_{2L}/ξ_L as a function of ξ_L/L for lattice sizes with fixed S_2 , and we compare them with Eq. (7). We stress that the theoretical curve is not a fit to the data: there is no free parameter to be chosen! Even if the agreement is not perfect, we notice that the points closer to the theoretical curve correspond to larger lattices.

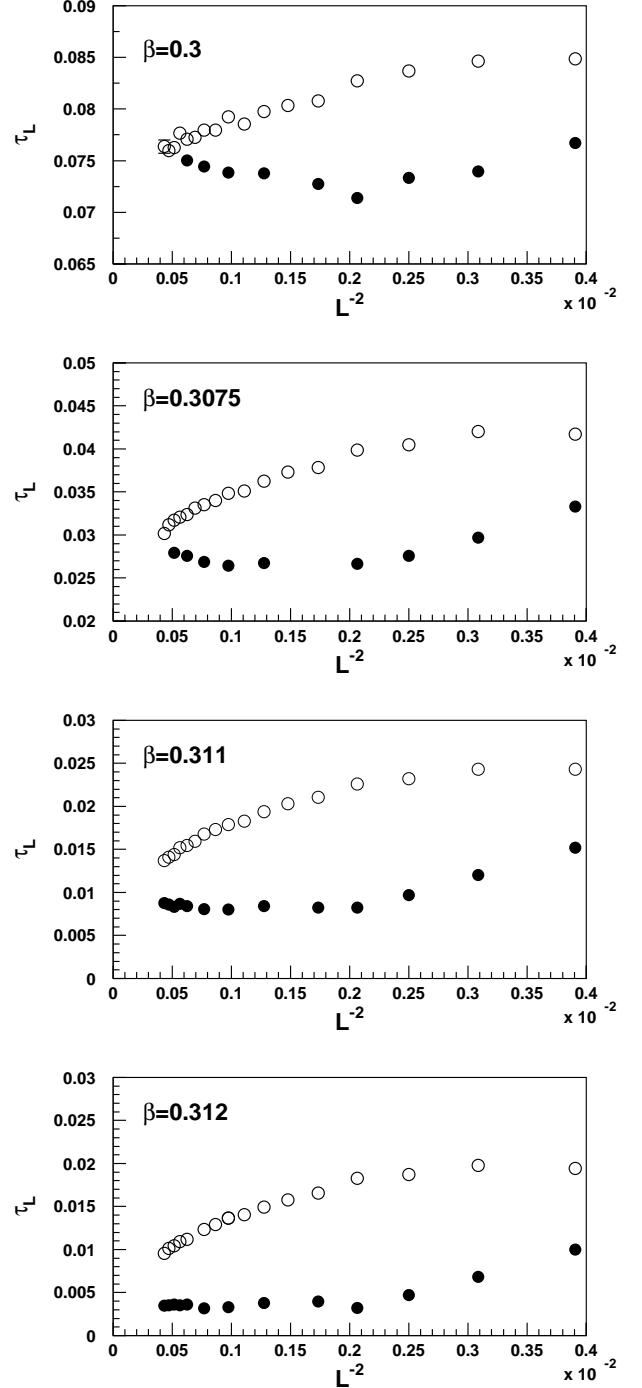


FIG. 1: τ_L for different inverse temperatures β . Filled (respectively empty) points refer to geometries with aspect ratio S_2 (respectively S_1) fixed. Here $S_2 \approx 0.200$, $S_1 \approx 0.106$. Errors are smaller than the size of the points.

Using the universal function $F_\xi(z, S_2, 2)$, we can extrapolate our data to infinite volume using the general strategy of Ref. [13]. Correspondingly, we obtain $\beta_c = 0.31256(9)$ and verify that, for small t , $\xi_\infty \sim t^{-1/2}$ as predicted by JSLC.

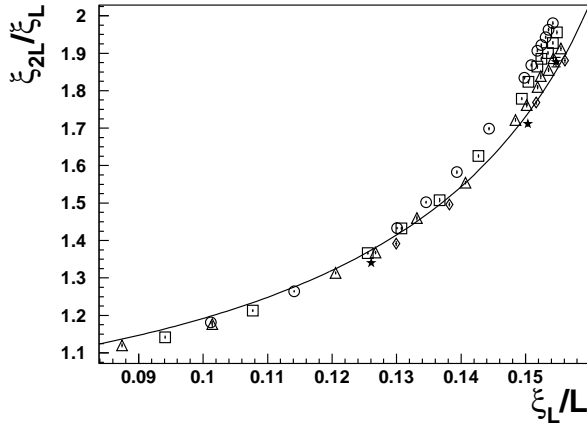


FIG. 2: FSS curve for the transverse correlation length at fixed $S_2 \approx 0.200$. Different symbols correspond to different lattice sizes: $L = 16(\circ)$, $18(\square)$, $20(\triangle)$, $22(\diamond)$, $24(\star)$. The solid curve is the function $F_\xi(z, S_2, 2)$ defined in Eq. (7).

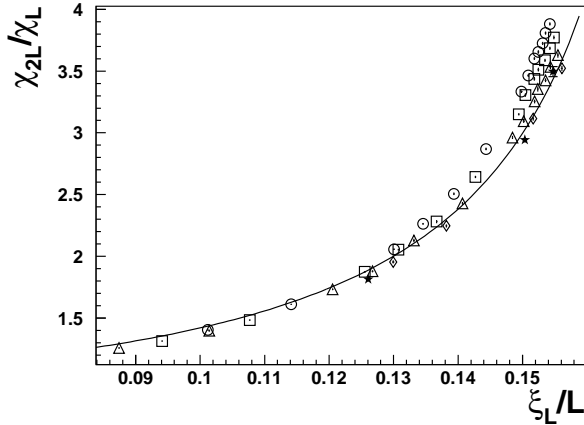


FIG. 3: FSS curve for the transverse susceptibility at fixed $S_2 \approx 0.200$. We use the same symbols of Fig. 2. The solid curve is the function $F_\chi(z, S_2, 2) = F_\xi(z, S_2, 2)^2$, where $F_\xi(z, S_2, 2)$ is defined in Eq. (7).

We can perform the same test for the susceptibility. In Fig. 3 we report our numerical results for χ_L together with the theoretical JSLC prediction. We observe a good agreement between theory and Monte Carlo results.

We have also measured the transverse Binder cumulant g_L , see Ref. [3] for the definition. At the critical point we observe that $g_L(\beta_c) \sim L^{-0.2}$ for $L \rightarrow \infty$. Thus, the Binder cumulant vanishes at the critical point, again in agreement with the idea that transverse fluctuations are Gaussian.

Further details of our analysis are presented elsewhere [14].

In conclusion, we have shown by means of a new FSS analysis and new Monte Carlo data that the critical behavior of transverse fluctuations in the DLG model is Gaussian. Indeed, the numerical data are in per-

fect agreement with Gaussian FSS functions. Note that our results are stronger than those presented in previous analyses. First, we check not only critical exponents but a full scaling function; second, in our comparisons there are no free parameters that can be tuned. Our results are in perfect agreement with the predictions of the JSLC model.

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